$\boxed{1.2.31}$

a) Soit $u \in A$. $u = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\1\end{pmatrix}$

Soit $v \in B$. $v = a\begin{pmatrix}1\\1\\1\end{pmatrix}$

Vu que $\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}$, $B \subset A$

$\Rightarrow A + B \subset A$ et $\dim(A) = 2$, ce qui fait que $A + B \neq \mathbb{R}^3$.

Déterminons $A \cap B$, qui doit être égal à B!

$$w = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\1\end{pmatrix} = a\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 0 & -1\\0 & 1 & -1\\0 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\a\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & -1\\0 & 1 & -1\\0 & 1 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -1\\0 & 1 & -1\\0 & 0 & 0\end{pmatrix} \Rightarrow \begin{matrix}x = a\\y = a\\a = a\end{matrix}$$

$$\Rightarrow A \cap B = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle = B$$

$\boxed{1.2.31}_2$

b) $A+B = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$\Rightarrow \dim(A+B) = 3 \Rightarrow A+B = \mathbb{R}^3$

$A \cap B: \begin{pmatrix} x \\ y \\ y \end{pmatrix} = \begin{pmatrix} a \\ a \\ -a \end{pmatrix} \Rightarrow \begin{matrix} x = a \\ y = a \\ y = -a \end{matrix}$

$\Rightarrow a = -a \Rightarrow a = 0 \Rightarrow x = y = 0$

$\Rightarrow A \cap B = \{0\} \Rightarrow \mathbb{R}^3 = A \oplus B$, la

somme est directe.

$\boxed{1.2.31}_3$

c) $A+B = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle$

$$\begin{pmatrix}1&0&0\\0&1&-1\\1&2&0\\0&0&1\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&-1\\0&2&0\\0&0&1\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$\Rightarrow A+B = \mathbb{R}^3$

Voyons si la somme est directe. $A \cap B$:

$$\begin{pmatrix}x\\y\\-y\end{pmatrix} = \begin{pmatrix}a\\2a\\b\end{pmatrix} \Leftrightarrow \begin{cases} x = a \\ y = 2a \\ y = -b \end{cases}$$

$\Leftrightarrow$ $x = a$ $\Rightarrow$ $w \in A \cap B$ ssi

$y = 2a$ $\quad w = a\begin{pmatrix}1\\2\\-2\end{pmatrix}$

$b = -2a$

$a = a$ $\quad \Rightarrow A \cap B = \left\langle \begin{pmatrix}1\\2\\-2\end{pmatrix} \right\rangle$

La somme n'est pas directe