$$\int \frac{3x-1}{x^2-x+1}\,dx = \frac{3}{2}\int \frac{2x-\frac{2}{3}}{x^2-x+1}\,dx$$

$\Delta = (-1)^2 - 4\cdot 1\cdot 1 = 1-4 = -5 < 0$

$$= \frac{3}{2}\int \frac{2x-1+\frac{1}{3}}{x^2-x+1}\,dx$$

$$= \frac{3}{2}\int \left(\frac{2x-1}{x^2-x+1} + \frac{1/3}{x^2-x+1}\right) dx$$

$$= \frac{3}{2}\left(\int \frac{2x-1}{x^2-x+1}\,dx + \int \frac{1/3}{x^2-x+1}\,dx\right)$$

$$= \frac{3}{2}\ln\left|x^2-x+1\right| + \frac{1}{2}\int \frac{1}{x^2-x+1}\,dx$$

Pour calculer la 2ème intégrale, on « complète le carré ».

$$\frac{1}{x^2-x+1} = \frac{1}{x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}$$

$$= \frac{1}{\left(x-\frac{1}{2}\right)^2+1-\frac{1}{4}} = \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

Soit maintenant le changement de variable suivant : $\left(x-\frac{1}{2}\right)^2 = \frac{3}{4}t^2$

$$\Leftrightarrow \quad x-\frac{1}{2} = \frac{\sqrt{3}}{2}t \quad \Leftrightarrow \quad t = \frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)$$

$$\Leftrightarrow \quad x = \frac{\sqrt{3}}{2}t+1 \quad \Rightarrow \quad dx = \frac{\sqrt{3}}{2}dt$$

On a donc

$$\int \frac{1}{x^2-x+1}\,dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \int \frac{1}{\frac{3}{4}t^2+\frac{3}{4}} \cdot \frac{\sqrt{3}}{2}\,dt$$

$$= \int \frac{4}{3} \cdot \frac{1}{t^2+1} \cdot \frac{\sqrt{3}}{2}\,dt = \frac{2\sqrt{3}}{3} \arctan(t)$$

$$= \frac{2\sqrt{3}}{3} \arctan\left(\frac{2x-1}{\sqrt{3}}\right)$$

Finalement,

$$\int \frac{3x-1}{x^2-x+1}\,dx = \frac{3}{2}\ln\left|x^2-x+1\right| + \frac{1}{2}\cdot\frac{2\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$= \frac{3}{2}\ln\left|x^2-x+1\right| + \frac{\sqrt{3}}{3}\cdot\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

<u>N.B.</u> $\frac{\sqrt{3}}{3} = \frac{\sqrt{3}\cdot\sqrt{3}}{3\cdot\sqrt{3}}$

$= \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$ (cf. corrigé du fascicule)