On suppose connues les égalités ci-dessous :

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad (1)$$

$$\sin(x+y) = \cos x \sin y + \cos y \sin x \qquad (2)$$

$$\sin^2 x + \cos^2 x = 1 \qquad (3)$$

$\boxed{\text{Égalité 1}}$ $\quad \cos(3\alpha) = 4\cos^3(\alpha) - 3\cos\alpha$

preuve : $\cos(2\alpha) = \cos(\alpha+\alpha) \overset{(1)}{=} \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$= \cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1-\cos^2\alpha)$

$= 2\cos^2\alpha - 1 \qquad (*)$

$\sin(2\alpha) = \sin(\alpha+\alpha) \overset{(2)}{=} \cos\alpha\sin\alpha + \cos\alpha\sin\alpha$

$= 2\sin\alpha\cos\alpha \qquad (**)$

On en déduit :

$$\cos(3\alpha) = \cos(2\alpha+\alpha) \overset{(1)}{=} \cos(2\alpha)\cos\alpha - \sin(2\alpha)\sin\alpha$$

$$\overset{(*)/(**)}{=} \left(2\cos^2\alpha - 1\right)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha$$

$$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$$

$$\overset{(3)}{=} 2\cos^3\alpha - \cos\alpha - 2\left(1-\cos^2\alpha\right)\cos\alpha$$

$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$$

$$= 4\cos^3\alpha - 3\cos\alpha$$

$\square$

Egalité 2 $\quad \sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha$

<u>preuve</u> : $\sin(3\alpha) = \sin(2\alpha+\alpha)$

$$= \cos(2\alpha)\sin\alpha + \cos\alpha\sin(2\alpha)$$

$$\overset{(*)/(**)}{=} \left(2\cos^2\alpha - 1\right)\sin\alpha + \underbrace{\cos\alpha\, 2\sin\alpha\cos\alpha}_{2\sin\alpha\cdot\cos^2\alpha}$$

$$\overset{(3)}{=} \left[2\left(1-\sin^2\alpha\right)-1\right]\sin\alpha + 2\sin\alpha\left(1-\sin^2\alpha\right)$$

$$= \left(1-2\sin^2\alpha\right)\sin\alpha + 2\sin\alpha - 2\sin^3\alpha$$

$$= 3\sin\alpha - 4\sin^3\alpha$$

$\square$

$\boxed{\text{Egalité 3}}$ $$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$

preuve:

$$\cos(2x) = \cos(x+x)$$

$$\overset{(1)}{=} \cos x\cos x - \sin x\sin x$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos^2 x - \left(1-\cos^2 x\right)$$

$$= 2\cos^2 x - 1$$

$$\Rightarrow \quad 2\cos^2 x - 1 = \cos(2x)$$

$$\Leftrightarrow \quad 2\cos^2 x = \cos(2x) + 1$$

$$\Leftrightarrow \quad \cos^2 x = \frac{\cos(2x)+1}{2}$$

On pose $x = \frac{\alpha}{2} \Leftrightarrow 2x = \alpha$ et on en déduit :

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{\cos\alpha + 1}{2} = \frac{1+\cos\alpha}{2} \quad \square$$

$\boxed{\text{Egalité 4}}$ $\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$

preuve:

$$\begin{aligned} \cos(2x) &= \cos(x+x) \\ &\overset{(1)}{=} \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x \\ &= 1 - \sin^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

$$\Rightarrow 1 - 2\sin^2 x = \cos(2x)$$

$$\Leftrightarrow 1 - \cos(2x) = 2\sin^2 x$$

$$\Leftrightarrow \sin^2 x = \frac{1-\cos(2x)}{2}$$

On pose $x = \frac{\alpha}{2} \Leftrightarrow 2x = \alpha$ et on en déduit:

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{2}$$

$\square$

$\boxed{\text{Egalité 5}}$ $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

preuve: $\cos(x-y) \overset{(1)}{=} \cos(x+(-y))$

$= \cos x \cos(-y) - \sin x \sin(-y)$

cercle trigo.

$= \cos x \cos y + \sin x \sin y$

$\Rightarrow \cos(x+y) + \cos(x-y) \overset{(1)}{=} (\cos x \cos y - \sin x \sin y)$

$+ (\cos x \cos y + \sin x \sin y)$

$= 2\cos x \cos y$

On pose $x = \frac{\alpha+\beta}{2}$ et $y = \frac{\alpha-\beta}{2}$

On en déduit :

$$x+y = \frac{\alpha+\cancel{\beta}+\alpha-\cancel{\beta}}{2} = \frac{2\alpha}{2} = \alpha$$

$$x-y = \frac{\alpha+\beta-(\alpha-\beta)}{2} = \frac{2\beta}{2} = \beta$$

Ce qui permet de conclure.

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$$

$$\Leftrightarrow \cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \quad \square$$

$\boxed{\text{Egalité 6}}$ $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

preuve : $\sin(x-y) = \sin(x+(-y))$

$$\overset{(2)}{=} \cos x \sin(-y) + \cos(-y)\sin x$$

$$= -\cos x \sin y + \cos y \sin x$$

$$\Rightarrow \sin(x+y) - \sin(x-y) \overset{(2)}{=} (\cos x \sin y + \cos y \sin x)$$
$$-(-\cos x \sin y + \cos y \sin x)$$

$$= 2\cos x \sin y$$

On pose $x = \dfrac{\alpha+\beta}{2}$ et $y = \dfrac{\alpha-\beta}{2}$.

On en déduit

$$x+y = \frac{\alpha+\beta+\alpha-\beta}{2} = \frac{2\alpha}{2} = \alpha$$

$$x-y = \frac{\alpha+\beta-(\alpha-\beta)}{2} = \frac{2\beta}{2} = \beta$$

$$\Rightarrow \sin(x+y) - \sin(x-y) = 2\cos x \sin y$$

$$\Leftrightarrow \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \quad \square$$